arXiv:hep-th/9412172v1 19 Dec 1994

On the stability
of p-brane

P.Demkin [1]

Institute of Theoretical Physics
Uppsala University
Box 803, S-75108
Uppsala, Sweden
paul@rhea.teorfys.uu.se

Abstract

Stability of some solutions of the equations of motion of bosonic
p-branes in curved and flat spacetimes is stated.

PACS Nos.: 03.70, 11.17.

---

[1]On leave from Department of Physics, Vilnius University, Saulėtekio al.9, 2054,
Vilnius, Lithuania

1 Introduction

Besides strings as one-dimensional extended relativistic objects, we may also consider p-branes as p-dimensional surfaces moving in spacetime. Such a membrane model naturally appears (i) when generalizing the known shell-electron model, which has already been suggested by Dirac [1, 2, 3] ; (ii) as a cosmic domain wall in the post-inflationary universe [4, 5] ; (iii) as an effective model of supergravity [6] ; and (iv) as, like superstring, a model unifying fundamental interactions [7, 8].

Let us turn to the last point. Unlike the properties of strings, those of p-branes are much less investigated so far [10, 9]. The quantum properties of supermembrane are known only on the semiclassical level. The spectrum continuity of supermembrane has been treated as its quantum and even classical instability, as its degenerate turning into an infinite string without changing its energy [11, 12] or else as its total instability. Therefore, the term "instability" in this context means asymptotical behaviour of p-brane solution at $t \to \infty$.

The existence of stable p-brane solution is important to the general theory of extended objects. For further development of the theory (quantization, perturbation theory and so on) we have to be sure that there is at least one example of stable solution of the equation of motion.

This paper shows that the class of stable p-brane solutions is not empty. For this purpose, we shall consider some of the solutions of p-brane equations of motion, both new and known ones [13, 15], in curved and flat spacetimes.

It is necessary to agree upon the main term 'stability' in advance. There are many kinds of stability that are known in mathematics (structural, Poisson, Lagrange, conditional, absolute *etc.*). Solutions of equations of motion are their stable points respecting their mappings. We shall restrict our consideration only to the asymptotical behaviour of the p-brane solution at $t \to \infty$ and to Lyapunov and asymptotical stabilities. The stable point x_0 of the mapping A is Lyapunov stable (and, respectively, asymptotically stable), if $\forall \varepsilon > 0$, so that if $|x - x_0| < \delta$, then $|A^n x - A^n x_0| < \varepsilon$ for all $0 < n < \infty$ (correspondingly, $A^n x - A^n x_0 \to 0$, as $n \to \infty$).

2 Bosonic membrane in curved spacetime

We may separate the bosonic part of the membrane in $D = 11$ by extinguishing fermionic degrees of freedom of the supermembrane. The action for the supermembrane becomes the action for a purely bosonic membrane in a curved spacetime:

$$S = -\frac{T}{2} \int d^3\xi \sqrt{h} [h^{ij} \partial_i x^M \partial_j x^N g_{MN}(x) -$$
$$-\frac{1}{3} \varepsilon^{ijk} \partial_i x^M \partial_j x^N \partial_k x^P B_{MNP}(x) - 1], \tag{1}$$

where $h_{ij} = \partial_i x^M \partial_j x^N g_{MN}$, $h = |det h_{ij}|$, $g_{MN}(x)$ is a metric of curved spacetime, $B_{MNP}(x)$ is an antisymmetric tensor field of rank 3, which couples with membrane via a Wess-Zumino term [7, 8, 9].

In this case, the equations of motion for membrane turn into

$$\partial_i (\sqrt{h} h^{ij} \partial_j x^N g_{MN}) - \frac{1}{2} \sqrt{h} h^{ij} \partial_i x^N \partial_j x^P \partial_N g_{MP} -$$
$$-\frac{1}{6} \varepsilon^{ijh} \partial_i x^N \partial_j x^P \partial_k x^Q F_{MNPQ} = 0, \tag{2}$$

where $F_{MNPQ} = 4\partial_{[M} B_{NPQ]}$ is the tension of potential B_{MNP}.

Let us consider the solution for the bosonic membrane in $D = 11$ with a special spherical spacetime symmetry metric $g_{MN}(x)$

$$ds^2 = -e^{2a} dt^2 + e^{2b}[dr^2 + r^2 d\Omega^2] + e^{2c}[dy + g\cos\theta d\varphi - qdt]^2 +$$
$$+ (dx^5)^2 + ... + (dx^{10})^2 \tag{3}$$

in the form

$$\xi^0 = t, \ \xi^1 = r, \ \xi^2 = y; \quad X^3 = \theta, \ X^4 = \varphi, \ X^N \neq f(\xi), \ N = 5, ..., 10. \tag{4}$$

Using the explicit expression for metric (3) and choosing coordinates (4) we obtain the equations of motion for the parameters a, b, c, q:

$$(a'' + \frac{2}{r} a')e^{-2b} - \frac{1}{2} q'^2 e^{-4(a+b)} = 0, \tag{5}$$

$$(a'^2 + b'^2 + \frac{1}{r} b' + a'b')e^{-2b} - \frac{1}{4} q'^2 e^{-4(a+b)} = 0, \tag{6}$$

2

$$(b'' + \frac{2}{r}b')e^{-2b} + \frac{g^2}{2r^4}e^{-2a-6b} = 0, \tag{7}$$

$$\left[a'' + b'' + \frac{2}{r}(a' + b')\right]e^{-2b} - \frac{1}{2}q'^2e^{-4(a+b)} + \frac{g^2}{2r^4}e^{-2a-6b} = 0, \tag{8}$$

$$q'' + (\frac{2}{r} - 4a' - 2b')q'e^{-2a-3b} = 0 \tag{9}$$

with the extra condition $a + b + c = const.$

The solutions to these equations are:

1. The stable monopole solution found by D.J.Gross and M.J.Perry [16]:

$$exp(2b) = exp(-2c) = 1 + \bar{g}/r, \quad a = 0, \quad q = 0, \quad g^2 = \bar{g}^2. \tag{10}$$

2. Its electrically charged analog

$$exp(-2a) = exp(2c) = 1 + \bar{A}/r, \quad b = 0,$$
$$q = -(A/r)(1 + \bar{A}/r)^{-1}, \quad g = 0, \quad A^2 = \bar{A}^2. \tag{11}$$

3. Its dyon analog

$$exp(-2a) = exp(2b) = (1 + \bar{g}/r)^2, \quad c = 0,$$
$$q = -(g/r)(1 + \bar{g}/r)^{-1}, \quad g^2 = 2\bar{g}^2. \tag{12}$$

When considering a purely bosonic membrane, in the supersymmetric action we can extinguish fermionic degrees of freedom, *i.e.* spacetime gravitino $\psi_M(X)$ and fermionic coordinates $\theta(\xi)$, or else let them equal zero. In this case we obtain a bosonic sector of the supermembrane in a curved spacetime. We have just considered such a way and obtained stable monopol-like solutions. The Nambu-Goto action in a curved spacetime is another way to consider a purely bosonic membrane. We may start from the Nambu-Goto action

$$S_D = -T \int d^{p+1}\xi\sqrt{|det\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}|}, \tag{13}$$

where $g_{\mu\nu} = g_{\mu\nu}(X)$.

The equations of motion derived from this action have the following form:

$$\partial_\alpha\left(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta x^\nu g_{\mu\nu}\right) = \frac{1}{2}\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\lambda\frac{\partial g_{\nu\lambda}}{\partial X^\mu}, \tag{14}$$

3

where $\gamma_{\alpha\beta}$ is completely defined by the induced metric

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \tag{15}$$

Let us consider the simplest spherically simmetric non-flat metric, $i.e.$ the Schwarzschild solution of the metric

$$g_{\mu\nu} = \begin{pmatrix} 1-q & 0 & 0 & 0 \\ 0 & -(1-q)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{16}$$

where $q = 2GM/r$, G is Newton's constant, and M is the total gravitational mass of the membrane.

For the $p = 2$ membrane in the spherical coordinate system in $D = 4$, $X^\mu = (\tau, r(\tau), \theta, \phi)$, the equation of motion (14) becomes

$$(1-q)r\ddot{r} - 2(1-q)\dot{r}^2 + \frac{1}{4}q\dot{r}^3 + \frac{1}{4}q(1-q)^2\dot{r} + 2(1-q)^3 = 0. \tag{17}$$

At $q = 1$ it gets into the black-hole region. In the case when $q \leq 1$ (close to unity), we may decompose the solution $r(\tau)$ near the Schwarzschild radius $r_0 = 2GM (r = r_0 + \rho)$. Then the equations of motion (17) turn into

$$\left[(r_0 + \rho)\rho\ddot{\rho}/r_0 + \frac{1}{4}\dot{\rho}^3 - 2\rho\dot{\rho}^2/r_0 \right](1 + \rho/r_0)^{-1} + \tag{18}$$

$$\left(\frac{1}{4}\rho^2\dot{\rho}/r_0^2 + 2\rho^3/r_0^3 \right)(1 + \rho/r_0)^{-3} = 0.$$

We may consider this equation accounting for the terms that make the greatest contribution into the evolution of $\rho = \rho(\tau)$:

$$\rho\ddot{\rho} + 2\rho^3/r_0^3 = 0. \tag{19}$$

The solution of this equation is

$$\rho(\tau) = \left[\rho^{-\frac{1}{2}}(0) + \frac{1}{\sqrt{3}r_0^{\frac{3}{2}}}(\tau - \tau_0) \right]^{-2}. \tag{20}$$

The velocity of such membrane motion is

$$\dot{\rho}(\tau) = -\frac{2}{\sqrt{3}}(\rho/r_0)^{\frac{3}{2}}, \tag{21}$$

$i.e.$ we obtain an asymptotical fall of the membrane on the Schwarzschild sphere during the infinite time.

3 Bosonic p-brane in flat spacetime

The equations of motion for bosonic relativistic p-brane in a flat spacetime are

$$\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\mu) = 0. \tag{22}$$

These equations and the constraint conditions

$$P_\tau^\mu X_{\mu;i} = 0, \qquad P^2 + T^2 det h_{ij} = 0, \tag{23}$$

where $P_\tau^\mu = \delta\mathcal{L}/\delta\dot{X}^\mu$, $\quad 1 \le i, j \le p$, $\quad$ and the border conditions for open dimensions

$$\frac{\partial h(\sigma_i^{in})}{\partial X_{;i}^\mu} = \frac{\partial h(\sigma_i^f)}{\partial X_{;i}^\mu} = 0, \qquad \sigma_i^\in \left[\sigma_i^{in}, \sigma_i^f\right] \tag{24}$$

may be obtained from the action

$$S = -T \int d^{p+1}\zeta \sqrt{|det(\partial_\alpha X^\mu \partial_\beta X_\mu)|}. \tag{25}$$

Equations of motion (22) are equivalent to the ordinary Laplace operator $\triangle$ on a non-compact manifold Σ^{p+1}, *i.e.* on the worldvolume of the p-brane. We know that the Laplace operator in the Euclidean spacetime is a stable operator under small perturbations near the solution X_0^μ as a linear elliptic operator. However, the worldvolume metric $h_{\alpha\beta}$ is a pseudo-Euclidean metric, and the stability in this case is yet unclear.

In order to have the physical picture of surface Σ^{p+1}, we may choose $X^0 = \tau$ and obtain

$$X^\mu = (\tau, X^m(\tau, \sigma_1, ..., \sigma_p)), \qquad m = 1, ..., D - 1. \tag{26}$$

Then the metric $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 - 1 & \dot{X}^m \partial_a X_m \\ \dot{X}^m \partial_a X_m & \partial_a X^m \partial_b X_m \end{pmatrix}. \tag{27}$$

Without any restriction we may use the reparametrization invariance of the action and suggest $h_{0\beta} = \dot{X}^m \partial_b X_m = 0$. Then the metric tensor is

$$h_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 - 1 & 0 \\ 0 & \partial_a X^m \partial_b X_m \end{pmatrix}. \tag{28}$$

The equations of motion become

$$\ddot{X}^m + \frac{1}{2}\partial_a \dot{X}^2 h^{ab}\partial_b X^m + (\dot{X}^2 - 1)\triangle X^m = 0, \tag{29}$$

$$\partial_\tau \left(\frac{\sqrt{h}}{\sqrt{1 - \dot{X}^2}} \right) = 0, \qquad \dot{X}^2 \leq 1, \tag{30}$$

where $\triangle = \frac{1}{\sqrt{h}}\partial_a(\sqrt{h}h^{ab}\partial_b)$ is the Laplace operator on the space part of the metric tensor h_{ab}. Now that we have defined the Laplace operator
on an appropriate Euclidean subspace, we may treat
these equations of motion as a general dynamic system.

Let us introduce new variables $Y^m = \dot{X}^m$ and consider small variations $\delta X^m = \xi^m$ and $\delta Y^m = \eta^m$ respecting the solution X_0^m, Y_0^m. For these variations we have

$$\dot{\xi}^m = \eta^m, \tag{31}$$

$$\dot{\eta}^m = \frac{\partial Q^m(X_0, Y_0)}{\partial X^n}\xi^n + \frac{\partial Q^m(X_0, Y_0)}{\partial Y^n}\eta^n, \tag{32}$$

where $Q^m = \frac{1}{2}\partial_a Y^2 h^{ab}\partial_b X^m$.

At $D = 1$, the behaviour of the dynamical system is known very well. In this case, the solutions of the characteristical equation $\lambda^2 - Q'_X(X_0, Y_0) = 0$ are purely imaginary, and this means that the point (X_0, Y_0) is either the centre or the focus. A necessary and sufficient condition for the existence of the centre is that the initial dynamical system must have the time-independent, real and holomorphous integral $F(X, Y) = C$ in the neighbourhood of the point (X_0, Y_0). In this case, the point (X_0, Y_0) corresponds to the minimum of the potential energy of the system. It seems likely that in an analytical system with D degrees of freedom, the solution of the equations of motion, which is not a minimum point, is unstable. It is a pity that this has not been proved for $D \geq 2$.

Let us consider a special solution to the equations of motion for a closed surface that is a mixture of pulsation and rigid rotation:

$$X^m(\tau, \sigma_1, ..., \sigma_p) = x(\tau)(\cos \varphi(\tau)n^k, \sin \varphi(\tau)n^l, 0, ..., 0), \tag{33}$$

where $n^k, n^l = (n_1, ..., n_d)$ is a d-dimensional unit vector describing the embedding of a p-dimensional closed surface in S^{d-1} and $d \leq (D-1)/2$.

In this case, equations (30) correspond to

$$\dot{x}^2 + x^2\dot{\varphi}^2 + \left(\frac{x}{C}\right)^{2p} = 1, \tag{34}$$

where C is a constant with the length dimension allowing to bring the equation for x into the form

$$\ddot{x} - x^{2p-2}/C^{2p}\triangle x = 0. \tag{35}$$

$\triangle = \frac{1}{\sqrt{\bar{g}}}\partial_a(\sqrt{\bar{g}}g^{ab}\partial_b)$ is the covariant Laplacian on the p-surface swept out by $\vec{n}$, and $g_{ab} = \partial_a n \partial_b n$ is the corresponding metric.

Let $L = x^2\dot{\varphi} = const$ corresponding to the conservation of the angular momentum. Then the equations of motion turn into

$$(\triangle + p)n^k = 0, \tag{36}$$
$$\ddot{x} - L^2/x^3 + px^{2p-1}/C^{2p} = 0. \tag{37}$$

The first equation is Laplace operator in Euclidean space R^d. If the metric of the surface is known, this is an ordinary linear operator on a compact manifold, and it is stable. In the general case, this is the equation of minimal surface in the Euclidean space. From the general theory of minimal surfaces we know that solutions to this equation do exist, and some of the solutions n_0^k are stable under small variations near n_0^k [17]. The second equation is an ordinary differential equation equivalent to

$$y = \dot{x}, \tag{38}$$
$$\dot{y} = f(x), \quad f(x) = L^2/x^3 - px^{2p-1}/C^{2p}, \quad f'(x_0) < 0. \tag{39}$$

The characteristical equation $\lambda^2 - f'(x_0) = 0$ has purely imaginary solutions. The energy conservation law in the form

$$\left(\frac{x}{C}\right)^{2p} = 1 - \dot{x}^2 - \frac{L^2}{x^2} \geq 0 \tag{40}$$

following from (34) is the holomorphous integral $F(X,Y) = C$ in the neighbourhood of the point (X_0, Y_0), $i.e.$ the necessary and sufficient condition for the centre to exist. Its Poincaré index equals to unity.

Equation of motion (37) for the radial part $x(\tau)$ depends on two parameters, but in fact we may neglect one of them. Condition (40) is valid for both every time moment τ and initial conditions. From (40), it follows that

$$\mathbf{y}^2 = 1 - k\mathbf{x}^{2p} - \frac{1}{\mathbf{x}^2}, \tag{41}$$

where $\mathbf{x} = \frac{x}{L}$ is the new dimensionless variable, $\mathbf{y} = \dot{\mathbf{x}}$, $k = \left(\frac{L}{x_0}\right)^{2p} \times$
$\times \left(1 - \dot{x}_0^2 - \frac{L^2}{x_0^2}\right)$. The phase diagrams of this equation are represented in Fig.1.

4 Discussion

In the general case, the last word concerning stability belongs to the behaviour of the second variation of the action in the solution point. However, when the solution is known, we may check its stability "by hand" considering small perturbations, as it has been done above.

Thus, we have stable solutions for bosonic p-brane in curved and flat spacetime. All of them obey the constraints (23). They prove the existence of stable solutions for further development of the theory of relativistic extended objects.

Now one may get the impression that author claims all known p-branes to be stable and compact. This is not so. For example, the elegant solution of equations (29)-(30), found by Kikkawa-Yamasaki [14] and generalized by Hoppe [15],

$$X(\tau, \vec{\sigma}) = (f_1(\vec{\sigma})\vec{n}_1(\tau), ..., f_p(\vec{\tau})\vec{n}_p(\tau), 0, ..., 0), \tag{42}$$

where $\vec{n}_r(\tau) = (cos\omega_r\tau, sin\omega_r\tau)$ and $f_p(\vec{\sigma})$ are arbitrary functions having to satisfy $\Sigma_{r-1}^p \omega_r^2 f_r^2(\vec{\sigma}) = 1$, is non-compact. In this case, indeed, it is possible to change the space configuration of the p-brane without changing its energy, because, according to the last condition, $f_r(\vec{\sigma}) \to \infty$ is possible at $\omega_r \to 0$.

It is very useful to compare the properties of the stability of solution (33) in Euclidean spacetime. In this case, equations of motion (29), (30) turn into equations

$$\ddot{X}^m + \frac{1}{2}\partial_a \dot{X}^2 h^{ab}\partial_b X^m + (\dot{X}^2 + 1)\triangle X^m = 0, \tag{43}$$

8

$$\partial_\tau \left(\frac{\sqrt{h}}{\sqrt{1 + \dot{X}^2}} \right) = 0, \qquad \dot{X}^2 \le 1, \qquad (44)$$

Substituting expression $X(\xi)$ (33) turns them into

$$(\triangle + p)n^k = 0, \qquad (45)$$
$$\ddot{x} - L^2/x^3 - px^{2p-1}/C^{2p} = 0. \qquad (46)$$

The first equation is the same as (36) in Minkowski spacetime, *i.e.* the equation of the minimal surface for the space part of the worldvolume metric, but equation (46) is of another character. The new variables $\mathbf{x}, \mathbf{y}$, like in the case of Minkowsi spacetime, turn equation (41) into

$$\mathbf{y}^2 = -1 + k\mathbf{x}^{2p} - \frac{1}{\mathbf{x}^2}. \qquad (47)$$

The phase diagrams of this equation are represented in Fig.2. These diagrams correspond to unstable solutions of the equation of motion with the potential

$$f(x) = \frac{L^2}{x^3} + p\frac{x^{2p-1}}{C^{2p}}, \qquad (48)$$

where $\mathbf{x} = \frac{x}{L}$ is a new dimensionless variable, $\mathbf{y} = \dot{\mathbf{x}}$, $k = \left(\frac{L}{x_0}\right)^{2p} \times$ $\times \left(1 + \dot{x}_0^2 + \frac{L^2}{x_0^2}\right)$, and all the solutions are non-compact for any $p \in \mathbf{Z}_+$ and $k > 0$.

Stability is only one among many other classical requirements imposed on p-branes. For example, border conditions (24) are very hard restrictions for open p-brane. In this connection we can mention that the solution of the equation of motion for cylindrical membrane found in [18] does not obey the border condition. Any physically appropriate solution must have no singularity. The equations of motion for spherical bosonic membrane (17) at $q = 0$ in the form

$$r\ddot{r} - (D-2)r\dot{r}^2 + (D-2) = 0, \qquad \dot{r}(r) = \frac{1}{r_0^{D-2}}\left[r_0^{2(D-2)} - r^{2(D-2)}\right]^{\frac{1}{2}} \quad (49)$$

have a peculiarity at $r = 0$: they collapse to zero.

As is known, stable classical solutions may be destroyed by quantum fluctuations. Thus, the next important question is the observance of the stability requirement on the quantum level.

9

Essential nonlinearity of the p-brane equations of motion makes it difficult to investigate their quantum stability. The solutions that are known to us have been considered only in the semiclassical approach [9, 19, 20], whereas the question of stability will be fully answered only if the quantum stability is proved beyond the frames of the perturbation theory.

5   Acknowledgements

References

[1] P.A.M.Dirac, Proc.R.Soc.(London)A 268 (1962) 57.

[2] D.H.Hartley, M.Önder and R.W.Tucker, Class.Quantum Grav. 6 (1989) 1301.

[3] A.O.Barut and M.Pavšič, Phys.Lett.B 306 (1993) 49.

[4] A.Linde and D.H.Lyth, Phys.Lett.B 246 (1990) 353.

[5] Z.Lalak, S.Thomas and B.A.Ovrut, Phys.Lett.B 306 (1993) 10.

[6] J.Hughes, J.Liu, J. Polchinski, Phys.Lett.B 180 (1986) 370.

[7] E.Bergshoeff, E.Sezgin and P.K.Townsend, Phys.Lett.B 180 (1986) 370.

[8] E.Bergshoeff, E.Sezgin and P.K.Townsend, Phys.Lett.B 189 (1987) 75.

[9] M.Duff, Class. Quantum Grav. 6 (1989) 1577.

[10] M.Duff, J.X.Lu, Nucl.Phys.B416(1994)301.

[11] B.de Wit, J.Hoppe and H.Nicolai, Nucl.Phys.B 305[FS23] (1988) 545.

[12] K.Fujikawa, J.Kubo, Nucl.Phys.B 356(1991)208.

[13] X.Li, Phys.Lett.B 205 (1988) 451.

[14] K.Kikkawa and M.Yamasaki, Progr.Theor. Phys. 76 (1986) 1379.

[15] J.Hoppe and H.Nicolai, Phys.Lett.B.196 (1987) 451.

[16] D.J.Gross and M.J.Perry, Nucl.Phys.B 226 (1983) 29.

[17] Dao Chong Thi, A.T.Fomenko, *Minimal surfaces and the Plateau problem*,(Moscow, Nauka, 1987),(In Russian).

[18] P.A.Collins and P.W.Tucker, Nucl.Phys.B 112 (1976) 150.

[19] P.Demkin, Some properties of the linearized model of (super)p- brane, UUITP 11/94.

[20] M.J.Duff, T.Inami, C.N.Pope, E.Sezgin, and K.S.Stelle, Nucl.Phys.B 297 (1988) 515.